Spectral gaps, log-concave perturbations of even measures

joint work with B. Klartag

F. Barthe

Institut de Mathématiques de Toulouse

Ctop 29-4-2021

Outline

Definitions and setting

- μ probability measure on $\mathbb{R}^n$
- For $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}_\mu(f) := \inf_{a\in\mathbb{R}} \int (f-a)^2 d\mu = \int \left(f - \int f\, d\mu\right)^2 d\mu.$$

Definition (Poincaré constant)

Let $C_P(\mu)$ be the least C s.t.

$$\mathrm{Var}_\mu(f) \le C \int |\nabla f|^2 d\mu,$$

for all f locally Lipschitz.

$C_P^{\mathrm{Class}}(\mu)$: the least C so that it holds for all f also in Class.

(PI) $\Longrightarrow$ Dimension-free concentration

- (Gromov-V. Milman). If $f : \mathbb{R}^n \to \mathbb{R}$ is 1-Lipschitz, then

$$\mu\left(\left\{x;\ \left|f(x) - \int f\,d\mu\right| > t\sqrt{C_P(\mu)}\right\}\right) \leq c_0 \exp(-c_1 t).$$

- Tensorisation property:

$$C_P(\mu_1 \otimes \cdots \otimes \mu_k) = \max_{1 \leq i \leq k} C_P(\mu_i).$$

- Example: $d\nu(t) := \frac{1}{2} e^{-|t|} dt, \quad t \in \mathbb{R},$

$$C_P(\nu) = 4$$

$$C_P(\nu^n) = 4$$

(PI) $\Longrightarrow$ Spectral gap

Set $d\mu(x) = e^{-V(x)}dx$ and $L = \Delta - \nabla V.\nabla = e^V \mathrm{div}(e^{-V}\nabla\ )$

- L is self-adjoint, non-positive

$$\int f\, Lg\, d\mu = -\int \nabla f.\nabla g\, d\mu$$

- $\ker(L) = \{constants\}$ and

$$\frac{1}{C_P(\mu)} = \inf\left\{\int(-Lf)f\, d\mu;\ \int f\, d\mu = 0 \text{ and } \int f^2 d\mu = 1\right\}.$$

Remark

A minimizer verifies $Lf = -\frac{1}{C_P(\mu)} f$ (+ Neumann if boundary).

Strictly convex potentials

$$d\mu(x) = e^{-V(x)} dx$$

Theorem (Brascamp-Lieb 1976)

If $\forall x,\ D^2 V(x) > 0,$

$$\mathrm{Var}_\mu(f) \leq \int \langle (D^2 V)^{-1} \nabla f, \nabla f \rangle \, d\mu$$

for all f loc. Lipschitz.

- If $D^2 V \geq \mathrm{Id}$ then $C_P(\mu) \leq 1$
- After approximation: $d\gamma_n(x) = e^{-|x|^2/2} \dfrac{dx}{(2\pi)^{n/2}}$
  If $d\mu = \rho d\gamma_n$ with ρ log-concave, then $C_P(\mu) \leq 1 = C_P(\gamma_n)$.
- "A log-concave perturbations improves the BL inequality"

Convex potentials/log-concave measures

Kannan-Lovász-Simonovits conjecture (KLS, 1995):

$d\mu(x) = e^{-V(x)}dx$, $V : \mathbb{R}^n \to (-\infty, +\infty]$ convex. Is it true that

$$C_P(\mu) \leq 1000\, C_P^{\mathrm{Lin}}(\mu) \text{ ???}$$

with

$$C_P^{\mathrm{Lin}}(\mu) = \sup_{\theta \in \mathbb{R}^n \setminus \{0\}} \frac{\mathrm{Var}_\mu(\langle \cdot, \theta \rangle)}{|\theta|^2} = \|\mathrm{Cov}(\mu)\|_{op}$$

Known general results:

- $C_P(\mu) \leq c\,\mathrm{Tr}(\mathrm{Cov}(\mu)) \leq c\, n\, \|\mathrm{Cov}(\mu)\|_{op}$ (KLS, Bobkov)
- $C_P(\mu) \leq c\, e^{c\sqrt{\log(n)\log\log(n)}}\, C_P^{\mathrm{Lin}}(\mu)$ (Chen 2020, after Eldan 2013)

More specific results about KLS

Approximate bounds:

- If $V(x) = V(|x_1|, \dots, |x_n|)$, then (Klartag, 2009) $C_P(\mu) \le c \log(1+n)^2 \, C_P^{\mathrm{Lin}}(\mu)$
- More general symmetries (B.-Cordero 2013)

Conjecture confirmed for $\lambda_K(dx) = 1_K(x) \frac{dx}{\mathrm{Vol}(K)}$, for $K =$

- ℓ_p^n-balls (Sodin 2008, Latała-Wojtaszczyk 2008)
- Simplices (B.-Wolff 2009)
- Bodies of revolution (Huet 2011)
- Some Orlicz Balls (Kolesnikov-E. Milman 2018, B.-Wolff 2019)
- perturbations, products$\cdots$

Central sections of ℓ_p^n balls

- For $K \subset \mathbb{R}^n$, with $\mathrm{vol}(K) \in (0, +\infty)$,

$$d\lambda_K(x) := \frac{1_K(x)}{\mathrm{vol}(K)} dx.$$

- $B_p^n := \left\{x \in \mathbb{R}^n;\ \sum_i |x_i|^p \leq 1\right\}$.

Theorem (B.-Klartag 2019)

Let $p \in [1, 2]$, and $E \subset \mathbb{R}^n$ a linear subspace, $d := \dim(E) \in [n/2, n]$. Then

$$C_P(\lambda_{B_p^n \cap E}) \leq c \log(1 + d)^2\, C_P^{\mathrm{Lin}}(\lambda_{B_p^n \cap E}).$$

Reduction to perturbed product measures

Theorem (Kolesnikov-E. Milman)

Let $d\mu(x) = e^{-V(x)}dx$ log-concave probability on $\mathbb{R}^d$, with $\min V = 0$.
There exists $t > 0$ s.t. $K = \{x \in \mathbb{R}^d;\ V(x) \leq t\}$ verifies

- $C_P^{\mathrm{Lin}}(\lambda_K) \geq c > 0$,
- $C_P(\lambda_K) \leq C \cdot C_P(\mu) \cdot \log\left(e + C_P(\mu)\sqrt{d}\right)$,

where C, c are universal constants.

Strategy:

- Apply to $d\mu(x) = \exp(-\|x\|_p^p)\frac{d^E x}{Z_E}$, probability on E.
- View μ as a limit of even log-concave perturbations of the product measure $\exp(-\sum_{i=1}^n |x_i|^p)\frac{dx}{Z_p^n}$.
- Bound their C_P from above.

Our new question

- $d\mu_i(t) = \varphi_i(t)\,dt$, $1 \leq i \leq n$ even log-concave on $\mathbb{R}$
- $\rho : \mathbb{R}^n \to \mathbb{R}^+$ even log-concave function s.t.

$$d\mu^{n,\rho}(x) = \rho(x) \prod_{i=1}^{n} d\mu_i(x_i)$$

is a probability measure on $\mathbb{R}^n$.

Is it true that

$$C_P(\mu^{n,\rho}) \leq \kappa C_P(\mu^{n,1}) = \kappa C_P(\mu_1 \otimes \cdots \otimes \mu_n) = \kappa \max_i C_P(\mu_i) \ ???$$

Remark

Question of Gadat and Panloup related to LASSO:
on $\mathbb{R}^n$, let $Q \geq 0$ a quadratic form and

$$d\mu(x) = \frac{1}{Z} e^{-\|x\|_1 - Q(x)}\,dx.$$

Is $C_P(\mu)$ upper-bounded uniformly in Q, in n?

Dimension 1: symmetry is necessary

Theorem (Bobkov, 1999)

Let $d\mu(t) = \varphi(t)\,dt$ a log-concave probability measure, then

$$C_P(\mu) \approx \mathrm{Var}(\mu) \approx \frac{1}{\varphi(\mathrm{median}(\mu))^2}.$$

Consider for $|a| < 1$

$$d\nu_a(t) = e^{-|t|+at}\frac{dt}{Z_a},$$

then $\lim_{a\to 1^-} C_P(\nu_a) = +\infty$.

Dimension 1: symmetry is sufficient

Classical facts: let μ be probability measure on $\mathbb{R}$ (no atoms)

- if μ is even, then $C_P(\mu) = C_P^{\mathrm{Odd}}(\mu)$.
- if $I \subset \mathbb{R}$ is an interval then $C_P(\mu_{|I}) \leq C_P(\mu)$;

Proposition (Roustant-B.-Iooss, 2017)

Let μ be even, and $\rho : \mathbb{R} \to \mathbb{R}$ be even and decreasing on $\mathbb{R}^+$, s.t. $\int \rho \, d\mu = 1$ then

$$C_P(\rho \, d\mu) \leq C_P(\mu).$$

Sketch of proof: For f odd,

- $\int f^2 \mathbf{1}_{(-a,a)} d\mu \leq C_P(\mu) \int (f')^2 \mathbf{1}_{(-a,a)} d\mu$
- $\rho(x) = \int_0^{+\infty} \mathbf{1}_{t \leq \rho(x)} dt = \int_0^{+\infty} \mathbf{1}_{\rho^{-1}([t,\infty))}(x) dt$

The worst case scenario

Theorem (B.-Klartag)

Assume μ_i and ρ as above (even, log-concave) and let

$$d\mu^{n,\rho}(x) = \rho(x) \prod_{i=1}^{n} d\mu_i(x_i)$$

Then

$$C_P(\mu^{n,\rho}) \leq c\, n\, C_P(\mu^{n,1}) = c\, n \max_i C_P(\mu_i).$$

Proof:

1. Check $\mathrm{Var}_{\mu^{n,\rho}}(\langle \cdot, e_i \rangle) \leq \mathrm{Var}(\mu_i)$.
2. By [KLS, Bobkov]

$$\begin{aligned} C_P(\mu^{n,\rho}) &\leq c\, \mathrm{Tr}(\mathrm{Cov}(\mu^{n,\rho})) \\ &\leq c \sum_{i \leq n} \mathrm{Var}(\mu_i) \leq c\, n \max_i C_P(\mu_i). \end{aligned}$$

$$\mathrm{Var}_{\mu^{n,\rho}}(\langle\cdot, e_i\rangle) = \int x_1^2\, \rho(x) \prod_i d\mu_i(x_i)$$

$$= \int_{\mathbb{R}} x_1^2 \underbrace{\Big(\int \rho(x_1, x_2, \ldots, x_n) \prod_{i\geq 2} d\mu_i(x_i) \Big)}_{g(x_1)} d\mu_1(x_1)$$

g is even, and log-concave, hence $\downarrow$ on $\mathbb{R}^+$. An increasing function and a decreasing function are negatively correlated

$$\mathrm{Var}_{\mu^{n,\rho}}(\langle\cdot, e_i\rangle) \leq \Big(\int x_1^2 d\mu_1(x_1) \Big) \Big(\underbrace{\int g(x_1) d\mu_1(x_1)}_{1} \Big) = \mathrm{Var}(\mu_1). \quad \square$$

???? When can we prove for all θ that:

$$\int \langle x, \theta\rangle^2 \rho(x) \prod_i d\mu_i(x_i) \leq \left(\int \langle x, \theta\rangle^2 \prod_i d\mu_i(x_i) \right) \underbrace{\int \rho(x) \prod_i d\mu_i(x_i)}_{1}$$

Gaussian mixtures (GM)

Definition

A probability measure ν on $\mathbb{R}$ is a Gaussian mixture if

$$\nu = \mathcal{L}aw(S \times G),$$

where $S > 0$ and $G \sim \mathcal{N}(0,1)$ are independent, that is

$$d\nu(x) = \left(\int_{(0,+\infty)} \frac{e^{-x^2/(2\sigma^2)}}{\sigma\sqrt{2\pi}} \, d\mathbb{P}_S(\sigma) \right) dx,$$

or $d\nu(x) = \psi(x^2)dx$ with ψ completely monotonic

Examples:

- Student distributions $G/\sqrt{Z/k}$ with $Z \sim \mathcal{X}^2(k)$.
- $d\nu(t) = \frac{1}{2}e^{-|t|}dt = \mathcal{L}aw(\sqrt{2\mathcal{E}} \times G)$ with $\mathcal{E} \sim \mathrm{Exp}(1)$.
- For $p \in (0,2]$, $d\nu_p(t) = \frac{1}{Z_p}e^{-|t|^p}dt$.

Extension of Hargé and Royen's Gaussian inequalities

Theorem (Eskenazis-Nayar-Tkocz, 2018)

Let $\mu = \mu_1 \otimes \cdots \otimes \mu_n$ with $\mu_1, \ldots, \mu_n$ Gaussian mixtures. Let $f, g : \mathbb{R}^n \to \mathbb{R}^+$ even and log-concave, then

$$\int fg \, d\mu \geq \left(\int f \, d\mu \right) \left(\int g \, d\mu \right).$$

If c is even convex, then $f = e^{-\varepsilon c}$ gives

$$\int cg \, d\mu \leq \left(\int c \, d\mu \right) \left(\int g \, d\mu \right).$$

If μ_i are GM and ρ is even log-concave,

$$\int \langle x, \theta\rangle^2 \rho(x) \prod_i d\mu_i(x_i) \leq \left(\int \langle x, \theta\rangle^2 \prod_i d\mu_i(x_i) \right) \int \rho(x) \prod_i d\mu_i(x_i)$$

Hence $\mathrm{Cov}(\mu^{n,\rho}) \leq \mathrm{Cov}(\mu_1 \otimes \cdots \otimes \mu_n) = \mathrm{Cov}(\mu^{n,1})$.

Best KLS bounds gives when μ_i log-concave:

$$\begin{aligned} C_P(\mu^{n,\rho}) &\leq \kappa_n \left\| \mathrm{Cov}(\mu_1 \otimes \cdots \otimes \mu_n) \right\|_{op} = \kappa_n \max_i \mathrm{Var}(\mu_i) \\ &\leq \kappa_n \max_i C_P(\mu_i) \end{aligned}$$

with $\kappa_n = e^{c\sqrt{\log(n) \log\log(n)}}$.

Theorem (B-Klartag, 2019)

If μ_i are log-concave GM and ρ is even log-concave, then

$$C_P(\mu^{n,\rho}) \leq c \log(1+n) \max_i C_P(\mu_i).$$

Tools for the log-concave case

Theorem (Klartag, 2009)

For μ log-concave and even, $C_P(\mu) = C_P^{\mathrm{Odd}}(\mu)$

Let $C_\infty(\mu)$ be minimal s.t. for all f,

$$\mathrm{Var}_\mu(f) \leq C_\infty(\mu) \| |\nabla f| \|_\infty^2 = C_\infty(\mu) \|f\|_{\mathrm{Lip}}^2$$

Theorem (E. Milman, 2009)

For μ log-concave, $C_P(\mu) \approx C_\infty(\mu)$. If ν is also log-concave,

$$d_{\mathrm{TV}}(\mu, \nu) \leq 1 - \varepsilon \Longrightarrow C_P(\nu) \leq \kappa(\varepsilon) C_P(\mu).$$

No dependence in the dimension!

Sketch of the proof:

Since $\mu^{n,\rho}$ is even and log-concave, enough to deal with f odd:

$$\begin{aligned}
\mathrm{Var}_{\mu^{n,\rho}}(f) &= \int f^2\, d\mu^{n,\rho} = \int f^2(x)\rho(x)\prod_{j=1}^{n} d\mu_i(x_i) \\
&= \int f^2(x)\rho(x)\prod_{j=1}^{n}\left(\int_{\mathbb{R}_+^*} \frac{e^{-\frac{x_j^2}{2\sigma_j^2}}}{\sigma_j\sqrt{2\pi}}\, dm_j(\sigma_j)\right) dx
\end{aligned}$$

$$= \int_{(\mathbb{R}_+^*)^n}\left(\int_{\mathbb{R}^n} f^2(x)\rho(x) e^{-\frac{1}{2}\sum_j \frac{x_j^2}{\sigma_j^2}} \frac{dx}{(2\pi)^{n/2}\prod_j \sigma_j}\right)\prod_{j=1}^{n} dm_j(\sigma_j)$$

Apply $\mathrm{Var}_\mu(f) \leq \int \langle (D^2V)^{-1}\nabla f, \nabla f\rangle\, d\mu$ to $d\mu(x) = e^{-V(x)}dx$,

$$D^2V(x) \geq \mathrm{Diag}\left(\frac{1}{\sigma_1^2}, \ldots, \frac{1}{\sigma_n^2}\right).$$

Set $d\mu_i(t) = \varphi_i(t)dt$.

$$\mathrm{Var}_{\mu^{n,\rho}}(f) \leq$$

$$\int \left(\int \left(\sum_{i=1}^{n} \sigma_i^2 (\partial_i f(x))^2 \right) \rho(x) e^{-\frac{1}{2}\sum_j \frac{x_j^2}{\sigma_j^2}} \frac{dx}{(2\pi)^{n/2} \prod_j \sigma_j} \right) \prod_{j=1}^{n} dm_j(\sigma_j)$$

$$= \sum_{i=1}^{n} \int_{\mathbb{R}^n} (\partial_i f(x))^2 \left(\int_{(\mathbb{R}_+^*)^n} \sigma_i^2 \prod_{j=1}^{n} \left(e^{-\frac{x_j^2}{2\sigma_j^2}} \frac{dm_j(\sigma_j)}{\sigma_j \sqrt{2\pi}} \right) \right) \rho(x)\, dx$$

$$= \sum_{i=1}^{n} \int_{\mathbb{R}^n} (\partial_i f(x))^2 \left(\int_{\mathbb{R}_+} \sigma_i e^{-\frac{x_i^2}{2\sigma_i^2}} \frac{dm_i(\sigma_i)}{\sqrt{2\pi}} \right) \prod_{j \neq i} \varphi_j(x_j)\, \rho(x)\, dx$$

$$= \sum_{i=1}^{n} \int_{\mathbb{R}^n} (\partial_i f(x))^2 \alpha_i(x_i)\, d\mu^{n,\rho}(x),$$

where

$$\alpha_i(t) := \frac{1}{\varphi_i(t)} \int_{\mathbb{R}_+^*} \sigma \frac{e^{-\frac{t^2}{2\sigma^2}}}{\sqrt{2\pi}} \, dm_i(\sigma) = \frac{1}{\varphi_i(t)} \int_{|t|}^{+\infty} u\varphi_i(u) \, du, \quad t \in \mathbb{R}.$$

Indeed:

$$\begin{aligned} \int_{|t|}^{+\infty} u\varphi_i(u)du &= \int_0^{+\infty} \left(\int_{|t|}^{+\infty} ue^{-\frac{u^2}{2\sigma^2}} du \right) \frac{dm_i(\sigma)}{\sigma\sqrt{2\pi}} \\ &= \int_0^{+\infty} \sigma^2 e^{-\frac{t^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{2\pi}} dm_i(\sigma). \end{aligned}$$

Lemma

Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be even and log-concave, s.t. $\int \varphi = 1$. Then for all $t \in \mathbb{R}$,

$$\frac{\int_{|t|}^{+\infty} u\varphi(u)du}{\varphi(t)} \leq \frac{|t|}{2\varphi(0)} + \frac{1}{4\varphi(0)^2},$$

with equality if $\exists \lambda > 0, \forall u, \varphi(u) = \lambda \exp(-\lambda|u|)/2$.

The final steps...

For f **odd**,

$$\mathrm{Var}_{\mu^{n,\rho}}(f) \leq \int \sum_i \left(\frac{|x_i|}{2\varphi_i(0)} + \frac{1}{4\varphi_i(0)^2} \right) (\partial_i f(x))^2 d\mu^{n,\rho}(x)$$

$$\leq \max_i \varphi_i(0)^{-2} \int \left(\max_i \left(|x_i| \frac{\varphi_i(0)}{2} \right) + \frac{1}{4} \right) |\nabla f(x)|^2 d\mu^{n,\rho}(x)$$

$$\leq \max_i \varphi_i(0)^{-2} \left(\int \left(\max_i \left(|x_i| \frac{\varphi_i(0)}{2} \right) + \frac{1}{4} \right) d\mu^{n,\rho}(x) \right) \| |\nabla f| \|_\infty^2$$

$$\underbrace{\leq}_{\text{claim}} \max_i C_P(\mu_i) \left(\frac{1}{4} + c \log(2n) \right) \| |\nabla f| \|_\infty^2 .$$

Milman's equivalence does not apply (it requires all f). A truncation argument by Klartag allows to conclude as if it did! □

Tying up loose ends (1)

- By the GM correlation inequality

$$\begin{aligned} & \int \max_i \left(|x_i| \varphi_i(0) \right) d\mu^{n,\rho}(x) \\ = \;& \int \max_i \left(|x_i| \varphi_i(0) \right) \rho(x) \prod d\mu_i(x_i) \\ \leq \;& \left(\int \max_i \left(|x_i| \varphi_i(0) \right) \prod d\mu_i(x_i) \right) \left(\int \rho(x) \prod d\mu_i(x_i) \right) \\ = \;& \mathbb{E} \max_i \left(|X_i| \varphi_i(0) \right) \end{aligned}$$

  where $X_i \sim \varphi_i(t)\, dt$ are log-concave.

- By Borell and Hensley $\|X_i\|_{\psi_1} \leq c\|X_i\|_2 \leq \frac{c}{\varphi_i(0)}$, hence

$$\mathbb{E} \max_{1 \leq i \leq n} \left(|X_i| \varphi_i(0) \right) \leq c \log(2n).$$

Tying up loose ends (2)

- We know: $\forall f$ odd, $\forall \mu^{n,\rho} \in \mathcal{C}$, $\mathrm{Var}_{\mu^{n,\rho}}(f) \leq \int \theta |\nabla f|^2 d\mu^{n,\rho}$, with $\theta \geq 0$ convex and even.
- Set $A := \{x;\ \theta(x) \leq 2 \int \theta \, d\mu^{n,\rho}\}$. It is convex, 0-symmetric and $\mu^{n,\rho}(A) \geq \frac{1}{2}$.
- $\mu^{n,\rho}_{|A} = \frac{\mathbf{1}_A}{\mu^{n,\rho}(A)} \mu^{n,\rho} =: \mu^{n,\widetilde{\rho}}$ is also in the class $\mathcal{C}$.
- $d_{TV}(\mu^{n,\rho}, \mu^{n,\widetilde{\rho}}) \leq \frac{1}{2}$, hence $C_P(\mu^{n,\rho}) \leq c\, C_P(\mu^{n,\widetilde{\rho}})$
- For f odd,

$$\begin{aligned}
\mathrm{Var}_{\mu^{n,\widetilde{\rho}}}(f) &\leq \int \theta |\nabla f|^2 d\mu^{n,\widetilde{\rho}} = \int_A \theta |\nabla f|^2 d\mu^{n,\widetilde{\rho}} \\
&\leq 2 \left(\int \theta d\mu^{n,\rho} \right) \int |\nabla f|^2 d\mu^{n,\widetilde{\rho}}
\end{aligned}$$

Hence $C_P(\mu^{n,\rho}) \leq c\, C_P(\mu^{n,\widetilde{\rho}}) \leq 2c \int \theta \, d\mu^{n,\rho}$. □

That’s all Folks!