Le deficit dans l'inégalité de Log-Sobolev gaussienne et la conjecture de Mahler

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Plan

Polar of convex bodies

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If K contains 0, then the bipolar theorem states that $K = (K^\circ)^\circ$.

So, in this case,

$$K = \{x \in \mathbf{R}^n : x \cdot y \leq 1, \forall y \in K^\circ\}.$$

(standardized way to write K as an intersection of half spaces).

Polar of convex bodies

Simple fact : The bigger is K, the smaller is K°.

For example, if $p, q \geq 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(B_p^n)^\circ = B_q^n,$$

where

$$B_p^n = \left\{ x \in \mathbf{R}^n : \sum_{i=1}^n |x_i|^p \leq 1 \right\}$$

denotes the ℓ_p unit ball of $\mathbf{R}^n$.

Santaló Inequality (1949)

Theorem

If $K \subset \mathbf{R}^n$ is a centrally symmetric convex body, then

$$\mathrm{Vol}(K)\mathrm{Vol}(K^\circ) \leq \mathrm{Vol}(B_2^n)^2,$$

where Vol denotes the Lebesgue measure on $\mathbf{R}^n$.

Inverse Santaló inequality : the Mahler conjecture

Conjecture (Mahler 1939)

1. If K is centrally symmetric, then

$$\mathrm{Vol}(K)\mathrm{Vol}(K^\circ) \geq \mathrm{Vol}(B_1^n)\mathrm{Vol}(B_\infty^n) = \frac{4^n}{n!}$$

2. For a general convex body K,

$$\mathrm{Vol}(K)\mathrm{Vol}(K^\circ) \geq \mathrm{Vol}(\Delta^n)\mathrm{Vol}((\Delta^n)^\circ) = \frac{(n+1)^{n+1}}{(n!)^2}$$

where Δ^n is any non-degenerate simplex of $\mathbf{R}^n$.

Known cases

1. Mahler proved the conjectures in dimension 2. Alternative proof by M. Meyer (1991).
2. Iriyeh and Shibata (2020) recently proved the conjecture (for symmetric convex bodies) in dimension 3. Alternative proof by M. Fradelizi, A. Hubard, M. Meyer, E. Roldan-Pensado and A. Zvavitch (preprint, 2019).
3. Saint-Raymond (1981) showed that the (symmetric) Mahler conjecture holds true for unconditional convex bodies, that is to say convex body K satisfying

$$x = (x_1, \ldots, x_n) \in K \Rightarrow (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \in K,$$

   for all $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$.
4. Bourgain and Milman (1987) showed that Mahler conjecture is asymptotically true : there exists some absolute constant $\alpha > 0$ such that for all $n \geq 1$ and all convex body $K \subset \mathbf{R}^n$, it holds

$$P(K) \geq \alpha^n P(\Delta^n).$$

Plan

Convex conjugation and polarity

The Fenchel-Legendre transform of $f : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ is the function denoted by f^* and defined by

$$f^*(y) = \sup_{x \in \mathbf{R}^n} \{x \cdot y - f(x)\}, \qquad y \in \mathbf{R}^n.$$

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Lemma

If $K \subset \mathbf{R}^n$ is a centrally symmetric convex body and $p, q \geq 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left(\frac{\| \cdot \|_K^p}{p} \right)^* = \frac{\| \cdot \|_{K^\circ}^q}{q},$$

where $\|x\|_K = \inf\{\lambda \geq 0 : x \in \lambda K\}$, $x \in \mathbf{R}^n$.

Functional form of the Santaló inequality

As proved by Ball (1987) in the case of even functions and then extended by Artstein-Avidan, Klartag and Milman (2004) and Fradelizi and Meyer (2007), the Santaló inequality admits the following functional form :

Theorem

For any measurable function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ such that $0 < \int e^{-f}\, dx < +\infty$, there exists $a \in \mathbf{R}^n$ such that

$$\int e^{-f_a}\, dx \int e^{-(f_a)^*}\, dx \leq (2\pi)^n,$$

where $f_a(x) = f(x-a)$, $x \in \mathbf{R}^n$. When f is even, a can be chosen to be 0.

More generally, if $\int x e^{-f(x)}\, dx = 0$, then a can be chosen to be 0 (Lehec 2009).

The functional form gives back the Santaló inequality

If K is symmetric, then applying the inequality to $f(x) = \frac{\|x\|_K^2}{2}$ for which $f^*(y) = \frac{\|y\|_{K^\circ}^2}{2}$ yields to

$$\int e^{-\frac{\|x\|_K^2}{2}}\,dx \int e^{-\frac{\|y\|_{K^\circ}^2}{2}}\,dy \leq (2\pi)^n.$$

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$$\int e^{-\frac{\|x\|_K^2}{2}}\,dx = \int_0^{+\infty} te^{-t^2/2}\mathrm{Vol}(\{x : \|x\|_K \leq t\})\,dt$$

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$$\mathrm{Vol}(K)\mathrm{Vol}(K^\circ) \leq \left(\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}\right)^2$$

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Functional form of the Mahler conjecture

Notation : We will denote by $\mathcal{F}(\mathbf{R}^n)$ the set of lower semi-continuous functions $f : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ which are convex and such that $f(x) < +\infty$ for at least one value of x.

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Definition

Let $c > 0$ and $n \geq 1$.

- We will say that that the functional inverse Santaló inequality $\mathrm{IS}_n(c)$ holds with the constant $c > 0$ if for all function $f \in \mathcal{F}(\mathbf{R}^n)$ such that $0 < \int e^{-f}\,dx$ and $0 < \int e^{-f^*}\,dx$, it holds

$$\int e^{-f}\,dx \int e^{-f^*}\,dx \geq c^n. \tag{1}$$

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- We will say that that the *symmetric* (resp. *unconditional*) functional inverse Santaló inequality $\mathrm{IS}_{n,s}(c)$ (resp. $\mathrm{IS}_{n,u}(c)$) holds with the constant $c > 0$ if (1) holds for all function $f \in \mathcal{F}_s(\mathbf{R}^n)$ (resp. $\mathcal{F}_u(\mathbf{R}^n)$) such that $0 < \int e^{-f}\,dx$ and $0 < \int e^{-f^*}\,dx$.

Functional form of the Mahler conjecture

The following result is due to Fradelizi and Meyer (2008) :

Theorem (Fradelizi-Meyer (2008))

The following are equivalent :

1. For all $n \geq 1$ and all symmetric (resp. unconditionnal) convex body $K \subset \mathbf{R}^n$,

$$\mathrm{Vol}(K)\mathrm{Vol}(K^\circ) \geq \frac{4^n}{n!}$$

2. For all $n \geq 1$, the inequality $\mathrm{IS}_{n,s}(4)$ (resp. $\mathrm{IS}_{n,u}(4)$) holds.

Note that the implication (2) $\Rightarrow$ (1) is true for fixed n.

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Theorem (Fradelizi-Meyer (2008))

1. The inequality $\mathrm{IS}_{n,u}(4)$ holds for all $n \geq 1$,
2. The inequality $\mathrm{IS}_1(e)$ holds true.

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If K is symmetric, then applying the inequality to $f(x) = \imath_K(x)$ (convex indicator of K) for which $f^*(y) = \|y\|_{K^\circ}$ yields to

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Plan

Transport and Entropy

General goal : Give an alternative form of reverse Santaló inequalities involving optimal transport and entropy functionals. These forms will be deduced from the functional inequalities $\mathrm{IS}_{n,s}$ using duality arguments.

The model case - Talagrand quadratic transport inequality

Definition

The quadratic optimal transport cost W_2^2 between two probability measures μ, ν on $\mathbf{R}^n$ is defined by

$$W_2^2(\nu,\mu) = \inf_\pi \iint |x-y|^2 \, \pi(dxdy)$$

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Definition

The relative entropy of a probability measure ν with respect to a measure m on $\mathbf{R}^n$ is defined by

$$H(\nu|m) = \int \log \frac{d\nu}{dm} \, d\nu$$

as soon as $\log^- \left(\frac{d\nu}{dm}\right) \frac{d\nu}{dm}$ is m-integrable.

Talagrand inequality and its dual form

Let γ_n be the standard Gaussian measure on $\mathbf{R}^n$.

Theorem (Talagrand (1996))

For all probability measure ν on $\mathbf{R}^n$, it holds $W_2^2(\nu, \gamma_n) \leq 2H(\nu|\gamma_n)$.

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The constant 2 is sharp.

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Theorem (Symmetric form of Talagrand inequality)

For all probability measures ν_1, ν_2 on $\mathbf{R}^n$, it holds

$$W_2^2(\nu_1, \nu_2) \leq 4H(\nu_1|\gamma_n) + 4H(\nu_2|\gamma_n),$$

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Démonstration.

Since W_2 is a distance,

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Equality cases : if $\nu_1 = \mathcal{N}(-a, 1)$ and $\nu_2 = \mathcal{N}(a, 1)$, $a > 0$, then $W_2^2(\nu_1, \nu_2) = 4a^2$ and $H(\nu_i|\gamma_n) = \frac{a^2}{2}$. □

Talagrand inequality and its dual form

Theorem (Bobkov-Götze (1999))

For a given probability measure μ on $\mathbf{R}^n$ the following properties are equivalent :

1. For all probability measures ν_1, ν_2 on $\mathbf{R}^n$, $W_2^2(\nu_1, \nu_2) \leq CH(\nu_1|\mu) + CH(\nu_2|\mu)$.
2. For all bounded continuous function f on $\mathbf{R}^n$,

$$\int e^{Q_C f}\, d\mu \int e^{-f}\, d\mu \leq 1,$$

where $Q_C f(x) = \inf_{y \in \mathbf{R}^n}\{f(y) + \frac{1}{C}|x-y|^2\}$, $x \in \mathbf{R}^n$.

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3. This alternative form was used by Bobkov, Gentil and Ledoux (2001) to give a proof of the Otto-Villani Theorem based on the Hamilton-Jacobi semigroup

$$Q_t f(x) = \inf_{y \in \mathbf{R}^n}\left\{f(y) + \frac{1}{t}|x-y|^2\right\}, x \in \mathbf{R}^n.$$

Talagrand inequality and its dual form

Bobkov and Götze result is based on the following classical duality relation between relative entropy and log-Laplace functionals ...

Proposition

Let m be a Borel measure on $\mathbf{R}^n$.
If ν is a probability measure on $\mathbf{R}^n$ with $\nu = h.m$ such that $\log h \in L^1(\nu)$, then

$$H(\nu|m) = \sup\left\{\int f\,d\nu - \log\int e^f\,dm : f \text{ s.t } \int e^f\,dm < +\infty\right\}.$$

Conversely, if $\int e^f\,dm < +\infty$, then

$$\log\int e^f\,dm = \sup\left\{\int f\,d\nu - H(\nu|m) : \nu = hm \text{ with } \log h \in L^1(\nu)\right\}.$$

Talagrand inequality and its dual form

Bobkov and Götze result is based on the following classical duality relation between relative entropy and log-Laplace functionals ...

Proposition

Let m be a Borel measure on $\mathbf{R}^n$.
If ν is a probability measure on $\mathbf{R}^n$ with $\nu = h.m$ such that $\log h \in L^1(\nu)$, then

$$H(\nu|m) = \sup \left\{ \int f \, d\nu - \log \int e^f \, dm : f \text{ s.t } \int e^f \, dm < +\infty \right\}.$$

Conversely, if $\int e^f \, dm < +\infty$, then

$$\log \int e^f \, dm = \sup \left\{ \int f \, d\nu - H(\nu|m) : \nu = hm \text{ with } \log h \in L^1(\nu) \right\}.$$

... and on the Kantorovich dual formulation of the W_2^2 transport cost :

Theorem

For all probability measures ν_1, ν_2 on $\mathbf{R}^n$,

$$W_2^2(\nu_1, \nu_2) = \sup_{f \in \mathcal{C}_b(\mathbf{R}^n)} \left\{ \int Q_1 f \, d\nu_1 - \int f \, d\nu_2 \right\}.$$

Proof of the Bobkov-Götze result

Suppose that

$$W_2^2(\nu_1, \nu_2) \leq CH(\nu_1|\mu) + CH(\nu_2|\mu), \qquad \forall \nu_1, \nu_2.$$

Proof of the Bobkov-Götze result

Suppose that

$$W_2^2(\nu_1,\nu_2) \leq CH(\nu_1|\mu) + CH(\nu_2|\mu), \qquad \forall \nu_1,\nu_2.$$

Then, using Kantorovich duality, we have that for all $g \in \mathcal{C}_b(\mathbf{R}^n)$, it holds

$$\int \frac{Q_1 g}{C}\, d\nu_1 - \int \frac{g}{C}\, d\nu_2 \leq H(\nu_1|\mu) + H(\nu_2|\mu), \qquad \forall \nu_1,\nu_2.$$

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So, using the duality between H and the log-Laplace transform, one gets

$$\log \int e^{\frac{Q_1 g}{C}}\, d\mu + \log \int e^{-\frac{g}{C}}\, d\mu \leq 0.$$

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The converse direction is similar.

Plan

Transport-Entropy form of the direct Santaló inequality

The following improvement of the symmetric Talagrand inequality is due to Max Fathi :

Theorem (Fathi (2018))

For all probability measures ν_1, ν_2 on $\mathbf{R}^n$, one of which is centered, it holds

$$W_2^2(\nu_1, \nu_2) \leq 2H(\nu_1|\gamma_n) + 2H(\nu_2|\gamma_n).$$

This result can be derived from Santaló inequality (and implies it back). So it can be considered as a transport-entropy form of the Santaló inequality.

Sketch of proof of Fathi's result

Recall the functional version of the direct Santaló inequality : for all measurable function $f : \mathbf{R}^n \to \mathbf{R}$ such that $\int x e^{-f(x)}\,dx = 0$

$$\int e^{-f}\,dx \int e^{-f^*(y)}\,dy \leq (2\pi)^n.$$

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Let $g : \mathbf{R}^n \to \mathbf{R}$ be a measurable function such that $\int x e^{-g(x)}\, \gamma_n(dx) = 0$. Applying the functional Santaló inequality to the function $f(x) = \frac{|x|^2}{2} + g(x)$, gives

$$\int e^{-g(x)}\, \gamma_n(dx) \int e^{-f^*(y) + \frac{|y|^2}{2}}\, \gamma_n(dy) \leq 1.$$

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$$\int e^{-g(x)}\,\gamma_n(dx) \int e^{-f^*(y)+\frac{|y|^2}{2}}\,\gamma_n(dy) \leq 1.$$

But, note that

$$-f^*(y) + \frac{|y|^2}{2} = \inf_{x\in\mathbf{R}^n}\left\{g(x) + \frac{|x|^2}{2} - x\cdot y\right\} + \frac{|y|^2}{2} = Q_2 g(y).$$

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Therefore

$$\int e^{Q_2 g}\,d\gamma_n \int e^{-g}\,d\gamma_n \leq 1, \qquad \forall g \text{ such that } \int xe^{-g(x)}\,\gamma_n(dx) = 0.$$

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Therefore

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$\rightsquigarrow$ Improved (τ) Property with constant 2 instead of 4.

Plan

Transport-Entropy forms of reverse Santaló inequalities

Notation : $\mathcal{P}_k(\mathbf{R}^n)$ is the set of probability measures on $\mathbf{R}^n$ with a finite moment of order $k \geq 1$.

Definition (Maximal correlation cost)

The maximal correlation transport cost $\mathcal{T}(\mu, \nu)$ between two probability measures $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^n)$ is defined by

$$\mathcal{T}(\mu, \nu) = \sup_{\pi} \iint x \cdot y \, \pi(dxdy)$$

where the supremum runs over the set of couplings of μ and ν.

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where the supremum runs over the set of couplings of μ and ν.

Proposition

For all $\mu, \nu \in \mathcal{P}_2(\mathbf{R}^n)$, it holds

$$\mathcal{T}(\mu, \nu) = \inf \left\{ \int f \, d\mu + \int f^* \, d\nu \right\},$$

where the infimum runs over the set of all $f \in \mathcal{F}(\mathbf{R}^n)$ (i.e lower semicontinuous and convex on $\mathbf{R}^n$).
Note that this dual definition makes sense on $\mathcal{P}_1(\mathbf{R}^n)$.

A first Transport-Entropy formulation of functional inverse Santaló inequalities

Theorem (G. 2020)

Let $c > 0$. The reverse Santaló inequality $\mathrm{IS}_n(c)$ holds if and only if

$$\inf_{\eta_1 \in \mathcal{P}_1(\mathbf{R}^n)} \{\mathcal{T}(\nu_1, \eta_1) + H(\eta_1|\mathrm{Leb})\} + \inf_{\eta_2 \in \mathcal{P}_1(\mathbf{R}^n)} \{\mathcal{T}(\nu_2, \eta_2) + H(\eta_2|\mathrm{Leb})\} \leq -n \log c + \mathcal{T}(\nu_1, \nu_2), \tag{2}$$

for all $\nu_1, \nu_2 \in \mathcal{P}_1(\mathbf{R}^n)$.

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for all $\nu_1, \nu_2 \in \mathcal{P}_1(\mathbf{R}^n)$.
In the case of the reverse Santaló inequality $\mathrm{IS}_{n,u}(c)$ (resp. $\mathrm{IS}_{n,s}(c)$), the same statement holds with the extra condition that $\nu_1, \nu_2, \eta_1, \eta_2$ belong to $\mathcal{P}_{u,1}(\mathbf{R}^n)$ (resp. $\mathcal{P}_{s,1}(\mathbf{R}^n)$).

Twisted Log-Laplace functional

We consider a twisted duality where the Log-Laplace functional

$$f \mapsto \Lambda(f|m) := \log \int e^{f}\, dm$$

is replaced by the functional $L(\,\cdot\,|m)$ defined by

$$L(f|m) := -\log \int e^{-f^*}\, dm, \qquad f \in \mathcal{F}(\mathbf{R}^n).$$

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Theorem (Cordero-Erausquin-Klartag)

If m is a log-concave measure on $\mathbf{R}^n$, then for all measurable functions $f_0, f_1 : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, it holds

$$\int e^{-((1-t)f_0+tf_1)^*}\, dm \geq \left(\int e^{-f_0^*}\, dm\right)^{1-t} \left(\int e^{-f_1^*}\, dm\right)^{t}.$$

In particular, the functional $L(\,\cdot\,|m)$ is convex on $\mathcal{F}(\mathbf{R}^n)$.

Proof of the convexity of $L(\cdot|m)$

Since m is log-concave, then as an immediate consequence of the Prekopa-Leindler inequality, it satisfies the following property : if $g_0, g_1, h : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ are measurable functions such that for some $t \in (0,1)$ it holds

$$h((1-t)x + ty) \leq (1-t)g_0(x) + tg_1(y), \qquad \forall x, y \in \mathbf{R}^n$$

then

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Note that if f_0, f_1 are two measurable functions, then

$$((1-t)f_0 + tf_1)^* \left((1-t)x + ty\right) \leq (1-t)f_0^*(x) + tf_1^*(y), \qquad \forall x, y \in \mathbf{R}^n, \qquad \forall t \in (0,1).$$

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So applying the inequality above to $g_0 = f_0^*$, $g_1 = f_1^*$ and $h = ((1-t)f_0 + tf_1)^*$ gives the result.

The dual of $L(\cdot|m)$

For $\nu \in \mathcal{P}_1(\mathbf{R}^n)$,

$$K(\nu|m) := \sup_{f \in L^1(\nu) \cap \mathcal{F}(\mathbf{R}^n)} \left\{ \int (-f)\, d\nu + \log \int e^{-f^*}\, dm \right\},$$

$$= \sup_{f \in L^1(\nu) \cap \mathcal{F}(\mathbf{R}^n)} \left\{ \int (-f)\, d\nu - L(f|m) \right\}.$$

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The functional $K(\cdot|m)$ admits a more pleasant alternative expression :

Proposition

For any $\nu \in \mathcal{P}_1(\mathbf{R}^n)$, it holds

$$K(\nu|m) = - \inf_{\eta \in \mathcal{P}_1(\mathbf{R}^n)} \{\mathcal{T}(\nu, \eta) + H(\eta|m)\},$$

and the infimum can be restricted to compactly supported η.

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and the infimum can be restricted to compactly supported η.
Moreover, if ν and m are symmetric (resp. unconditional), then the infimum can be restricted to (compactly supported) elements of $\mathcal{P}_{s,1}(\mathbf{R}^n)$ (resp. $\mathcal{P}_{u,1}(\mathbf{R}^n)$).

Sketch of proof

By definition, and using the usual duality between entropy and log-Laplace, one gets

$$K(\nu|m) = \sup_{f\in L^1(\nu)\cap\mathcal{F}(\mathbf{R}^n)} \left\{ \int (-f)\, d\nu + \log \int e^{-f^*}\, dm \right\}$$

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$$= \sup_{f \in L^{\mathbf{1}}(\nu) \cap \mathcal{F}(\mathbf{R}^n)} \sup_{\eta \in \mathcal{P}_{\mathbf{1}}(\mathbf{R}^n)} \left\{ \int (-f)\, d\nu + \int (-f^*)\, d\eta - H(\eta|m) \right\}$$

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Sketch of proof of the theorem

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So, if $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbf{R}^n)$, then

$$\int -f^*\,d\nu_1 + \log \int e^{-f}\,dx + \int -f\,d\nu_2 + \log \int e^{-f^*}\,dx \geq n \log c - \left(\int f^*\,d\nu_1 + \int f\,d\nu_2\right)$$

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So, by definition of $K(\,\cdot\,|\mathrm{Leb})$, it holds

$$K(\nu_1|\mathrm{Leb}) + K(\nu_2|\mathrm{Leb}) \geq n \log c - \left(\int f^*\, d\nu_1 + \int f\, d\nu_2 \right).$$

Optimizing over f gives the claim.

Sketch of proof of the theorem

For the other direction, one needs the following reverse duality expressing L as conjugate of K.

Theorem (G. 2020)

Suppose that m is an arbitrary absolutely continuous log-concave measure. For any $f \in \mathcal{F}(\mathbf{R}^n)$ such that $\int e^{-f^*}\,dm > 0$, it holds

$$\sup_{\nu \in \mathcal{P}_1(\mathbf{R}^n)} \left\{ \int (-f)\,d\nu - K(\nu|m) \right\} = L(f|m).$$

If m and f are further assumed to be unconditional (resp. symmetric), then the supremum above can be restricted to $\mathcal{P}_{u,1}(\mathbf{R}^n)$ (resp. $\mathcal{P}_{s,1}(\mathbf{R}^n)$).

Plan

Moment measures

Definition (Moment measure)

If $V \in \mathcal{F}(\mathbf{R}^n)$ is such that $0 < \int e^{-V} < +\infty$, the moment measure of V is the probability measure ν defined as the push forward of the probability measure $\eta(dx) = \frac{e^{-V(x)}}{\int e^{-V(y)}\,dy} dx$ under the map ∇V. By extension, we also say that ν is the moment measure of η.

Characterization of moment measures

Theorem (Cordero-Erausquin-Klartag/Santambrogio)

1. A probability measure $\nu \in \mathcal{P}(\mathbf{R}^n)$ is the moment measure of some log-concave probability measure η_o on $\mathbf{R}^n$ such that $\eta_o(dx) = e^{-V_o}\,dx$ for some essentially continuous convex function $V_o : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ if and only if $\nu \in \mathcal{P}_1(\mathbf{R}^n)$, ν is centered and its support is not contained in an hyperplan. The function V_o is moreover unique up to translations.

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2. If ν is centered and its support is not contained in an hyperplan, then the probability measure η_o is up to translations the unique minimizer of the functional $\eta \mapsto \mathcal{T}(\nu, \eta) + H(\eta|\mathrm{Leb})$ on $\mathcal{P}_1(\mathbf{R}^n)$:

$$\inf_{\eta \in \mathcal{P}_1(\mathbf{R}^n)} \{\mathcal{T}(\nu, \eta) + H(\eta|\mathrm{Leb})\} = \mathcal{T}(\nu, \eta_o) + H(\eta_o|\mathrm{Leb}).$$

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1. A probability measure $\nu \in \mathcal{P}(\mathbf{R}^n)$ is the moment measure of some log-concave probability measure η_o on $\mathbf{R}^n$ such that $\eta_o(dx) = e^{-V_o}\, dx$ for some essentially continuous convex function $V_o : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ if and only if $\nu \in \mathcal{P}_1(\mathbf{R}^n)$, ν is centered and its support is not contained in an hyperplan. The function V_o is moreover unique up to translations.
2. If ν is centered and its support is not contained in an hyperplan, then the probability measure η_o is up to translations the unique minimizer of the functional $\eta \mapsto \mathcal{T}(\nu, \eta) + H(\eta|\mathrm{Leb})$ on $\mathcal{P}_1(\mathbf{R}^n)$:

$$\inf_{\eta \in \mathcal{P}_1(\mathbf{R}^n)} \{\mathcal{T}(\nu, \eta) + H(\eta|\mathrm{Leb})\} = \mathcal{T}(\nu, \eta_o) + H(\eta_o|\mathrm{Leb}).$$

3. Moreover, if $\nu \in \mathcal{P}_{u,1}(\mathbf{R}^n)$ (resp. $\mathcal{P}_{s,1}(\mathbf{R}^n)$) then $\eta_o \in \mathcal{P}_{u,1}(\mathbf{R}^n)$ (resp. $\mathcal{P}_{s,1}(\mathbf{R}^n)$).

Characterization of moment measures

Theorem (Cordero-Erausquin-Klartag/Santambrogio)

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2. If ν is centered and its support is not contained in an hyperplan, then the probability measure η_o is up to translations the unique minimizer of the functional $\eta \mapsto \mathcal{T}(\nu,\eta) + H(\eta|\mathrm{Leb})$ on $\mathcal{P}_1(\mathbf{R}^n)$:
$$\inf_{\eta \in \mathcal{P}_1(\mathbf{R}^n)} \{\mathcal{T}(\nu,\eta) + H(\eta|\mathrm{Leb})\} = \mathcal{T}(\nu,\eta_o) + H(\eta_o|\mathrm{Leb}).$$
3. Moreover, if $\nu \in \mathcal{P}_{u,1}(\mathbf{R}^n)$ (resp. $\mathcal{P}_{s,1}(\mathbf{R}^n)$) then $\eta_o \in \mathcal{P}_{u,1}(\mathbf{R}^n)$ (resp. $\mathcal{P}_{s,1}(\mathbf{R}^n)$).

Definition

A function $V \in \mathcal{F}(\mathbf{R}^n)$ is said to be essentially continuous if the set of points were it is discontinuous (as a function taking values in $\mathbf{R} \cup \{\infty\}$) is negligible for the measure $\mathcal{H}_{n-1}$. Equivalently, V is essentially continuous if letting $D = \mathrm{dom}(V)$
$$\mathcal{H}_{n-1}\left(\{x \in \partial D : V(x) < \infty\}\right) = 0.$$
Note in particular that in dimension 1, a function $V \in \mathcal{F}(\mathbf{R})$ is essentially continuous if and only if it is continuous as a function taking values in $\mathbf{R} \cup \{+\infty\}$.

Some informal explanations

Let $\eta_0(dx) = h_0(x)\,dx$ be the minimizer of $\eta \mapsto \mathcal{T}(\nu, \eta) + H(\eta|\mathrm{Leb})$.

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If $\eta_t(dx) = ((1-t)h_0 + th)\,dx$ for some density h, then

$$\mathcal{T}(\nu, \eta_t) + H(\eta_t|\mathrm{Leb}) \geq \mathcal{T}(\nu, \eta_0) + H(\eta_0|\mathrm{Leb})$$

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$$\frac{H(\eta_t|\mathrm{Leb}) - H(\eta_0|\mathrm{Leb})}{t} \geq \frac{\mathcal{T}(\nu,\eta_0) - \mathcal{T}(\nu,\eta_t)}{t} \geq \int f_0(h_0 - h)\,dx$$

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So letting $t \to 0^+$, yields to

$$\int (\log h_0)(h - h_0) \geq \int f_0(h_0 - h)\,dx$$

So h_0 minimizes $h \mapsto \int (\log h_0 + f_0) h\,dx$ over densities.

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$$C_0 := \inf_{x\in\mathbf{R}^n}(\log h_0(x) + f_0(x)) = \int (\log h_0 + f_0)h_0\,dx$$

and so $\log h_0 + f_0 = C_0$ η_0 almost surely.

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Optimal transport theory : ∇f_0 is the Brenier map sending η_0 on ν.

Other equivalent forms of reverse Santaló inequalities

Theorem (G. 2020)

Let $c > 0$; the following propositions are equivalent :

1. Inequality $\mathrm{IS}_n(c)$ holds.
2. For all log-concave probability measures η_1, η_2 on $\mathbf{R}^n$ such that, for $i = 1, 2$, $\eta_i(dx) = e^{-V_i}\,dx$ for some essentially continuous convex function $V_i : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, it holds

$$(*) \qquad \mathcal{T}(\nu_1, \eta_1) + H(\eta_1|\mathrm{Leb}) + \mathcal{T}(\nu_2, \eta_2) + H(\eta_2|\mathrm{Leb}) \leq -n \log c + \mathcal{T}(\nu_1, \nu_2),$$

where ν_1, ν_2 are the moment measures of η_1 and η_2.

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   where ν_1, ν_2 are the moment measures of η_1 and η_2.

Moreover, if $V_i : \mathbf{R}^n \to \mathbf{R}$, then $(*)$ reduces to

$$H(\eta_1|\mathrm{Leb}) + H(\eta_2|\mathrm{Leb}) \leq -n \log(e^2 c) + \mathcal{T}(\nu_1, \nu_2).$$

The same result holds for inequality $\mathrm{IS}_{n,u}(c)$ (resp. $\mathrm{IS}_{n,s}(c)$) with the extra condition that η_1, η_2 are unconditional (resp. symmetric).

Comments

$$(*) \qquad \mathcal{T}(\nu_1,\eta_1) + H(\eta_1|\mathrm{Leb}) + \mathcal{T}(\nu_2,\eta_2) + H(\eta_2|\mathrm{Leb}) \leq -n\log c + \mathcal{T}(\nu_1,\nu_2),$$

Remark

- If ν is the moment measure of $\eta = e^{-V(x)}\,dx$, then

$$\mathcal{T}(\nu,\nu) = \int |x|^2\,d\nu = \int |\nabla V|^2 e^{-V}\,dx$$

So $(*)$ is some bivariate version of the Euclidean log-Sobolev inequality.

- Moreover if $V < +\infty$ on $\mathbf{R}^n$, then

$$\mathcal{T}(\nu,\eta) = \int x\cdot\nabla V(x)e^{-V(x)}\,dx = n \qquad \text{Integration by parts.}$$

Gaussian version

Definition

The Fisher information of $\eta(dx) = h(x)\,\gamma_n(dx)$ with respect to γ_n is defined by

$$I(\eta|\gamma_n) = 4\int |\nabla(h^{1/2})|^2\,\gamma_n(dx),$$

as soon as $h^{1/2}$ is almost surely differentiable on $\mathbf{R}^n$.

Remark : Usually, Fisher information is defined as above only for $h^{1/2} \in W^{1,2}(\gamma_n)$ (and $+\infty$ otherwise).

Theorem (G. 2020)

Let $c > 0$ and $n \geq 1$. The inverse functional Santaló inequality $\mathrm{IS}_n(c)$ holds if and only if for all log-concave probability measures η_1, η_2 on $\mathbf{R}^n$ such that, for $i = 1, 2$, $\eta_i(dx) = e^{-V_i}\,dx$ for some *essentially continuous* $V_i \in \mathcal{F}(\mathbf{R}^n)$, it holds

$$H(\eta_1|\gamma_n) + H(\eta_2|\gamma_n) + \frac{1}{2}W_2^2(\nu_1, \nu_2) \leq \frac{1}{2}I(\eta_1|\gamma_n) + \frac{1}{2}I(\eta_2|\gamma_n) + n\log(2\pi/c),$$

as soon as $\nu_1, \nu_2 \in \mathcal{P}_2(\mathbf{R}^n)$, where, for $i = 1, 2$, $\nu_i = \nabla(V_i)_{\#}\eta_i$ is the moment probability measure of η_i.
The same statement holds for $\mathrm{IS}_{n,s}(c)$ (resp. $\mathrm{IS}_{n,u}(c)$) with the extra condition that η_1, η_2 are symmetric (resp. unconditional).

A bound on the deficit in the unconditional case

Consider the following quantity

$$\delta_n(\eta) = \frac{1}{2} I(\eta|\gamma_n) - H(\eta|\gamma_n).$$

According to the Log-Sobolev Inequality for γ_n, it holds $\delta_n(\eta) \geq 0$, for any η with a regular density.

Corollary

For any log-concave and unconditional probability measure η on $\mathbf{R}^n$ with $\eta(dx) = e^{-V(x)}\,dx$ where $V : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is an essentially continuous convex function, it holds

$$H(\eta|\gamma_n) + \frac{1}{2} W_2^2(\nu, \lambda_{C_n}) \leq \frac{n}{2} \log\left(\frac{\pi e}{2}\right) + \frac{1}{2} I(\eta|\gamma_n),$$

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In other words, for such η,

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Moreover, there exists a sequence of product measures $(\alpha_k^{\otimes n})_{k \geq 1}$ such that

$$\delta_n(\alpha_k^{\otimes n}) - \frac{1}{2} W_2^2\left(\nu_k^{\otimes n}, \lambda_{C_n}\right) + \frac{n}{2} \log\left(\frac{\pi e}{2}\right) \to 0,$$

as $k \to \infty$, where for $k \geq 1$, $\nu_k^{\otimes n}$ denotes the moment measure of $\alpha_k^{\otimes n}$.

Proof

For all η_1, η_2 unconditional, log-concave with good densities, it holds

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Take $\eta_2(dx) = \frac{1}{2^n}e^{-\sum_{i=1}^n |x_i|}\, dx$ (case of equality in the functional version). The moment measure of η_2 is λ_{C_n} and

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Optimality : Consider the sequence of 1d probability measures $(\alpha_k)_{k\geq 1}$ given by $\alpha_k(dx) = \frac{1}{Z_k}e^{-V_k(x)}\,dx$, with

$$V_k(x) = \begin{cases} k|x-1| & \text{if } x \geq 1 \\ 0 & \text{if } x \in [-1,1] \\ k|x+1| & \text{if } x \leq -1 \end{cases} \qquad \text{and} \qquad Z_k = \frac{2(k+1)}{k}.$$

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When $k \to \infty$, α_k approximates the uniform measure on $[-1,1]$ (equality case in the functional version).

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Taking $\eta_1 = (\alpha_k)^{\otimes n}$, one sees that the difference between left and right hand sides of the inequality goes to 0 as $k \to +\infty$.

Plan

1. Santaló inequalities and their functional forms
   - Santaló inequality
   - Functional forms
2. Transport-entropy forms of direct and reverse Santaló inequalities
   - The model case - Talagrand quadratic transport inequality
   - Transport-Entropy form of the direct Santaló inequality
   - A first Transport-Entropy form of reverse Santaló inequalities
   - Moment measures and other equivalent forms of reverse Santaló inequality
3. Proofs in dimension 1
   - The symmetric case in dimension 1
   - The non-symmetric case

Proofs in dimension 1

Joint work with Simon Zugmeyer.

Plan

The symmetric case in dimension 1

Theorem (G. - Zugmeyer (2020))

If η_1, η_2 are symmetric log-concave probability measures on $\mathbf{R}$ with a continuous density, then

$$H(\eta_1 \mid \mathrm{Leb}) + H(\eta_2 \mid \mathrm{Leb}) \leq -2(\log 2 + 1) + \mathcal{T}(\nu_1, \nu_2)$$

where ν_1, ν_2 are the moment measures of η_1, η_2.

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   - Santaló inequality
   - Functional forms
2. Transport-entropy forms of direct and reverse Santaló inequalities
   - The model case - Talagrand quadratic transport inequality
   - Transport-Entropy form of the direct Santaló inequality
   - A first Transport-Entropy form of reverse Santaló inequalities
   - Moment measures and other equivalent forms of reverse Santaló inequality
3. Proofs in dimension 1
   - The symmetric case in dimension 1
   - The non-symmetric case

The non-symmetric case

Similar ideas work for the non-symmetric case in dimension one.

Theorem (G. - Zugmeyer (2020))

If η_1, η_2 are log-concave probability measures on $\mathbf{R}$ with a continuous density, then

$$H(\eta_1 \mid \mathrm{Leb}) + H(\eta_2 \mid \mathrm{Leb}) \leq -3 + \mathcal{T}(\nu_1, \nu_2)$$

where ν_1, ν_2 are the moment measures of η_1, η_2.

Open questions

1. How to recover the unconditionnal case in dimension n ?
   In dimension 2, some tensorization arguments with the Knothe coupling partially work but only give the result for unconditionnal convex potentials $V : \mathbf{R}^2 \to \mathbf{R}$ such that $\frac{\partial^2 V}{\partial x \partial y} \geq 0$ on $\mathbf{R}^2_+$.
2. How to recover the 2 dimensional transport-entropy version of Mahler conjecture ?
3. Is it possible to recover the Bourgain-Milman theorem using transport arguments ?